

Local geodesics for plurisubharmonic functions

Alexander Rashkovskii

Abstract

We study geodesics for plurisubharmonic functions from the Cegrell class \mathcal{F}_1 on a bounded hyperconvex domain of \mathbb{C}^n and show that, as in the case of metrics on Kähler compact manifolds, they linearize an energy functional. As a consequence, we get a uniqueness theorem for functions from \mathcal{F}_1 in terms of total masses of certain mixed Monge-Ampère currents. Geodesics of relative extremal functions are considered and a reverse Brunn-Minkowski inequality is proved for capacities of multiplicative combinations of multi-circled compact sets. We also show that functions with strong singularities generally cannot be connected by (sub)geodesic arcs.

1 Introduction

Starting with pioneer work by Mabuchi [19], a notion of geodesics in the space of Kähler metrics on compact complex manifolds has been playing a prominent role in Kähler geometry and has found a lot of applications. We will not give here any detailed account on this subject; the interested reader can consult, for example, [23], [10], [14], [1], [5], [15], and the bibliography therein. In particular, geodesics in the space of metrics on a compact n -dimensional Kähler manifold (X, ω) have been characterized as solutions to a complex homogeneous equation, which implies linearity of the Mabuchi functional

$$\mathcal{M}(\psi, \phi_0) = \frac{1}{n+1} \int_X (\psi - \phi_0) \sum_{k=0}^n (dd^c \psi)^k \wedge (dd^c \phi_0)^{n-k} \quad (1)$$

along the geodesics $\psi = \psi_t$ (here ϕ_0 is a reference metric).

We believe however that a local, flat situation of functions on a bounded pseudoconvex domain D of \mathbb{C}^n deserves independent consideration, at least because of possible applications. The simplest choice here are functions with zero boundary values on ∂D and finite total Monge-Ampère mass. To provide existence of the corresponding boundary problem on $D \times \{1 < |\zeta| < e\}$, we require also finiteness of the Monge-Ampère energy $\mathbf{E}(u) = \int_D u (dd^c u)^n$. For such (not necessarily bounded) plurisubharmonic functions we show in Theorem 5.2 that the energy functional $u \mapsto \mathbf{E}(u)$ plays role of the Mabuchi functional (1). We use this in proving a uniqueness result (Theorem 3.4 and Corollary 5.3) for functions from the Cegrell class $\mathcal{F}_1(D)$ in terms of total masses of $n+1$ mixed Monge-Ampère currents on D .

We discuss briefly geodesics connecting relative extremal functions ω_{K_j} of compact subsets K_j of D . In the multi-circled case, a variant of reversed Brunn-Minkowski inequality is proved for the Monge-Ampère capacities of multiplicative combinations of K_j . We present a simple example where the geodesic functions u_t are still relative extremal functions, however not of compact sets but of multi-plate condensers.

The case of bounded functions (Theorem 3.3) is close to the classical setting of Kähler metrics, with a modification to handle the boundary effects. The general case requires a

justification for existence of solutions of the corresponding boundary problem like that in [3] and [11]. We show that while this works for $\mathcal{F}_1(D)$ (Theorem 5.2), for functions with strong singularities (say, with positive Lelong numbers) such a problem generally has no solution (Theorem 6.2).

2 Energy functional on Cegrell classes

Let $D \subset \mathbb{C}^n$ be a bounded hyperconvex domain. We recall that Cegrell's class $\mathcal{E}_0(D)$ consists of bounded plurisubharmonic functions u in D with zero boundary values on ∂D and finite total Monge-Ampère mass

$$\int_D (dd^c u)^n < \infty;$$

class $\mathcal{E}_1(D)$ consists of functions u that are limits of decreasing sequences $u_j \in \mathcal{E}_0(D)$ such that

$$\sup_j \int_D |u_j| (dd^c u_j)^n < \infty;$$

if, in addition,

$$\sup_j \int_D (dd^c u_j)^n < \infty,$$

then $u \in \mathcal{F}_1(D)$.

If $u \in \mathcal{E}_1(D)$, then the current $(dd^c u)^n$ is defined as the limit of $(dd^c u_j)^n$ and is independent of the choice of the approximating sequence u_j [7, Thm. 3.8].

For any function $u \in \mathcal{E}_1(D)$, consider its energy functional

$$\mathbf{E}(u) = (n+1) \mathcal{M}(u, 0) = \int_D u (dd^c u)^n. \quad (2)$$

For any sequence u_j from the definition of $\mathcal{E}_1(D)$, we have $\mathbf{E}(u_j) \rightarrow \mathbf{E}(u)$ [7, Thm. 3.8].

Similarity with the Mabuchi functional (1) for metrics on compact manifolds becomes visible from the following important identity.

Proposition 2.1 *For any $u, v \in \mathcal{E}_1(D)$,*

$$\mathbf{E}(u) - \mathbf{E}(v) = \int_D (u - v) \sum_{k=0}^n (dd^c u)^k \wedge (dd^c v)^{n-k}. \quad (3)$$

Proof. This easily follows from the integration by parts formula

$$\int_D u dd^c v \wedge T = \int_D v dd^c u \wedge T \quad (4)$$

valid for $u, v \in \mathcal{E}_1$ and positive closed currents T [8, Cor. 3.4]. \square

Corollary 2.2 *If $u, v \in \mathcal{E}_1(D)$ satisfy $u \leq v$, then $\mathbf{E}(u) \leq \mathbf{E}(v)$. If, in addition, $u \in \mathcal{F}_1(D)$ and $\mathbf{E}(u) = \mathbf{E}(v)$, then $u = v$ on D .*

Proof. The inequality is well known (see, for example, [7, Thm. 3.8]) and follows, in particular, directly from Proposition 2.1.

The condition $\mathbf{E}(u) = \mathbf{E}(v)$ gives us, by (3), $(dd^c u)^n = 0$ on the set $A = \{z : u(z) < v(z)\}$. We claim that this implies $u = v$ everywhere in D . In [6], this was proved for locally bounded u and v ; we adapt the proof to our case. Let $P(z) = |z|^2 - C \in \text{PSH}^-(D)$. If $u(z_0) < v(z_0)$, then the set $A_\eta = \{z : u(z) < \eta P(z) + v(z)\}$ has positive Lebesgue measure for some $\eta > 0$.

By [7, Lemma 4.4],

$$\eta^n \int_{A_\eta} (dd^c P)^n \leq \int_{A_\eta} (dd^c(\eta P + v))^n \leq \int_{A_\eta} (dd^c u)^n \leq \int_{\{u < v\}} (dd^c u)^n = 0,$$

which contradicts the positivity of the Lebesgue measure of A_η . \square

Remark. The second statement of Corollary 2.2 remains true if the condition $u \in \mathcal{F}_1(D)$ is replaced by $u \in \mathcal{E}_1(D)$ and $\lim u(z) = 0$ as $z \rightarrow \partial D$. In this case (increasing, if needed, the constant C in the definition of the function P), the set A_η is compactly supported in D and thus both u and v have finite Monge-Ampère mass on a neighborhood of \overline{A}_η , so [7, Lemma 4.4] still can be applied.

3 Geodesics for the class \mathcal{E}_0

Let S be the annulus $\{\zeta \in \mathbb{C} : 1 < |\zeta| < e\}$ bounded by the circles $S_0 = \{|\zeta| = 1\}$ and $S_1 = \{|\zeta| = e\}$. Given two functions $u_0, u_1 \in \mathcal{E}_0(D)$, consider the class $W(u_1, u_2)$ of all functions $u \in \text{PSH}^-(D \times S)$ such that $\limsup u(z, \zeta) \leq u_j(z)$ as $\zeta \rightarrow S_j$. The class is not empty because, for example, it contains $u_0 + u_1$.

Denote

$$\widehat{u}(z, \zeta) = \sup\{u(z, \zeta) : u \in W(u_1, u_2)\}.$$

Since its u.s.c. regularization \widehat{u}^* belongs to $W(u_1, u_2)$, we have $\widehat{u} = \widehat{u}^*$. Moreover, being a maximal plurisubharmonic function, it satisfies the homogeneous Monge-Ampère equation

$$(dd^c \widehat{u})^{n+1} = 0 \text{ on } D \times S. \quad (5)$$

Evidently, $\widehat{u}(z, \zeta) = \widehat{u}(z, |\zeta|)$ on $D \times S$, so the function $u_t(z) := \widehat{u}(z, e^t)$ is convex in $t \in (0, 1)$; we will call it the *geodesic* of u_0 and u_1 . Similar to [5], we get

Proposition 3.1 *The geodesic u_t of $u_0, u_1 \in \mathcal{E}_0(D)$ has the following properties:*

- (i) $u_t(z) \rightarrow 0$ as $z \rightarrow \partial D$;
- (ii) $u_t \rightarrow u_j$ as $t \rightarrow j$, uniformly on D ($j = 0, 1$);
- (iii) $u_t \leq U_t := (1 - t)u_0 + tu_1$;
- (iv) $u_t \geq s_t := \max\{u_0 - M_1 t, u_1 - M_0(1 - t)\}$, where $M_j = \|u_j\|_\infty$.

Proof. Since $u_t \geq u_0 + u_1$, we have (i). Relation (iii) follows because $U_0 = u_0$, $U_1 = u_1$ and U_t is harmonic in t (while u_t is convex in t). The lower bound (iv) is evident because $\widehat{s}(z, \zeta) := s_{\log|\zeta|}(z)$ belongs to $W(u_0, u_1)$. Finally, (iii) and (iv) imply (ii). \square

A family of functions $v_t \in \mathcal{E}_0(D)$, $0 < t < 1$, will be called a *subgeodesic* for u_0 and u_1 if $\widehat{v}(z, \zeta) := v_{\log|\zeta|}(z) \in W(u_0, u_1)$.

Let us study values of the energy functional \mathbf{E} on curves in $\mathcal{E}_0(D)$. Here again we get its properties as in the case of compact manifolds.

Proposition 3.2 *The functional $v \mapsto \mathbf{E}(v)$ is concave on $\mathcal{E}_0(D)$.*

Proof. Let $U_t = (1-t)u_0 + tu_1$, $0 < t < 1$. By Proposition 2.1,

$$\frac{d}{dt} \mathbf{E}(U_t) = (n+1) \int_D (u_1 - u_0) (dd^c U_t)^n,$$

so

$$\begin{aligned} \frac{1}{n+1} \frac{d^2}{dt^2} \mathbf{E}(U_t) &= n \int_D (u_1 - u_0) \wedge dd^c(u_1 - u_0) \wedge (dd^c U_t)^{n-1} \\ &= -n \int_D d(u_1 - u_0) \wedge d^c(u_1 - u_0) \wedge (dd^c U_t)^{n-1} \leq 0, \end{aligned}$$

which proves the claim. \square

It also turns out that, on the other hand, the function $\mathbf{E}(v_t)$ is convex along subgeodesics.

Theorem 3.3 *Let v_t be a subgeodesic for $u_0, u_1 \in \mathcal{E}_0(D)$. Then the function $t \mapsto \mathbf{E}(v_t)$ is convex, and it is linear if and only if the subgeodesic v_t is a geodesic.*

Proof. The idea of the proof is similar to that for Proposition 3.2, however it needs more technicalities.

Convexity of $\mathbf{E}(v_t)$ is equivalent to subharmonicity of the function

$$\widehat{\mathbf{E}} = \mathbf{E}(\widehat{v}) = \int_D \widehat{v} (d_z d_z^c \widehat{v})^n,$$

and the linearity of \mathbf{E} corresponds to the harmonicity of $\widehat{\mathbf{E}}$. The corresponding result for the Mabuchi functional (1) on a compact manifold X follows from the formula

$$d_\zeta d_\zeta^c \widehat{\mathcal{E}} = \int_X (dd^c \widehat{v})^{n+1} \quad (6)$$

(see, for example, [1]), and one gets then the claims from the plurisubharmonicity of the subgeodesics and equation (5).

In the case of functions from $\mathcal{E}_0(D)$, $D \subset \mathbb{C}^n$, one can argue as follows. By [9, Thm. 1.2], \widehat{v} is the limit of a decreasing sequence of smooth functions $\widehat{v}^{(j)}$ from $\mathcal{E}_0(D \times S)$; clearly, they can be assumed to be independent of the argument of ζ . Furthermore, since $v_t^{(j)} \in \mathcal{E}_0(D)$

decrease to $v_t \in \mathcal{E}_0(D)$, we have $\mathbf{E}(v_t^{(j)}) \rightarrow \mathbf{E}(v_t)$ by [7, Thm. 3.8]. So, we can assume $\widehat{v} \in \mathcal{E}_0(D \times S) \cap C^\infty(D \times S)$.

Note that the aforementioned approximation theorem rests on the following result from [18], see also [9, Lem. 2.2]: *If $\varphi, \psi \in \text{PSH}(\Omega)$ and $b : \mathbb{R} \rightarrow \mathbb{R}_+$ is a smooth convex function with $b(x) = |x|$ for all $|x| > \epsilon > 0$, then $\max_b(\varphi, \psi) := \varphi + \psi + b(\varphi - \psi) \in \text{PSH}(\Omega)$.*

If we take here $\Omega = D \times S$, $\varphi = \widehat{v} - 2\epsilon$, and $\psi = \rho/\epsilon$ for a smooth exhaustion function ρ of D (which exists by [9, Cor. 1.3]), then $\max_b(\varphi, \psi) \in \mathcal{E}_0(D \times S) \cap C^\infty(D \times S)$. Moreover, it coincides with ρ/ϵ near $\partial D \times S$, so it is independent of ζ there. Since $\max_b(\varphi, \psi) \rightarrow \widehat{v}$ uniformly as $\epsilon \rightarrow 0$, we can thus also assume $d_\zeta \widehat{v} = 0$ near ∂D .

By Proposition 2.1,

$$d_\zeta^c \widehat{\mathbf{E}} = (n+1) \int_D d_\zeta^c \widehat{v} \wedge (d_z d_z^c \widehat{v})^n,$$

so

$$\begin{aligned} \frac{1}{n+1} d_\zeta d_\zeta^c \widehat{\mathbf{E}} &= \int_D d_\zeta d_\zeta^c \widehat{v} \wedge (d_z d_z^c \widehat{v})^n + n \int_D d_\zeta^c \widehat{v} \wedge d_\zeta (d_z d_z^c \widehat{v}) \wedge (d_z d_z^c \widehat{v})^{n-1} \\ &= \int_D d_\zeta d_\zeta^c \widehat{v} \wedge (d_z d_z^c \widehat{v})^n - n \int_D d_z d_\zeta^c \widehat{v} \wedge d_z^c d_\zeta \widehat{v} \wedge (d_z d_z^c \widehat{v})^{n-1} \\ &= \frac{1}{n+1} \int_D (dd^c \widehat{v})^{n+1}, \end{aligned}$$

where the second equality follows from Stokes' theorem because $d_\zeta \widehat{v} = 0$ near ∂D , and the last one by direct calculation with $d = d_z + d_\zeta$, $d^c = d_z^c + d_\zeta^c$.

Finally, let $v_j = \lim v_t$ as $t \rightarrow j$ for $j = 0, 1$, and let w_t be the geodesic of v_0, v_1 . If $\mathbf{E}(v_t)$ is linear, then $\mathbf{E}(v_t) = \mathbf{E}(w_t)$, so $v_t = w_t$ for all t by Corollary 2.2. \square

Now we can prove the following uniqueness result.

Theorem 3.4 *Let $u_0, u_1 \in \mathcal{E}_0(D)$ satisfy*

$$\int_D u_0 (dd^c u_0)^k \wedge (dd^c u_1)^{n-k} = \mathbf{E}(u_1), \quad k = 0, \dots, n. \quad (7)$$

Then $u_0 = u_1$ in D .

Proof. By (4), condition (7) implies

$$\int_D u_1 (dd^c u_0)^k \wedge (dd^c u_1)^{n-k} = \mathbf{E}(u_1), \quad k = 0, \dots, n,$$

as well, so

$$\int_D (u_1 - u_0) (dd^c u_0)^k \wedge (dd^c u_1)^{n-k} = 0, \quad k = 0, \dots, n. \quad (8)$$

Denote $U_t = (1-t)u_0 + tu_1$. By (8) and a computation in the proof of Proposition 3.2, the function $\mathbf{E}(U_t)$ is linear on $[0, 1]$, so $\mathbf{E}(U_t) = \mathbf{E}(u_0)$.

On the other hand, by Proposition 3.1, the geodesic u_t of u_0 and u_1 satisfies $u_t \leq U_t$ and, by Theorem 3.3, $\mathbf{E}(u_t) = \mathbf{E}(u_0)$ as well. By Corollary 2.2, we get $u_t = U_t$ for any t .

Therefore, the function $\widehat{U}(z, \zeta) = (1 - \log |\zeta|) u_0(z) + \log |\zeta| u_1(z)$ is plurisubharmonic in $D \times S$. Then

$$\frac{\partial}{\partial \bar{z}_k}(u_1 - u_0) = 0$$

for all k , so $u_1 - u_0$ is analytic in D , equal to 0 on ∂D , and thus is identical 0. \square

Remark. If $u \in \mathcal{E}_0(D)$ and $u_j = \max\{u, -\alpha_j\}$, then we have

$$\int_D (dd^c u_0)^k \wedge (dd^c u_1)^{n-k} = \int_D (dd^c u_1)^n, \quad k = 0, \dots, n,$$

for any $\alpha_0, \alpha_1 > 0$. Therefore, using the mixed energy functionals in Theorem 3.4 is essential.

4 Example: geodesics of relative extremal functions

Here we consider a particular case of the construction above. Recall that the *relative extremal function* of a set $K \Subset D$ is

$$\omega_K(z) = \limsup_{x \rightarrow z} \sup\{u(x) : u \in \text{PSH}^-(D), u|_K \leq -1\} \in \mathcal{E}_0(D).$$

We will be interested in the following: *Given two relatively compact subsets K_0 and K_1 of D , let $u_j = \omega_{K_j}$ for $j = 1, 2$, what can be said about their geodesic u_t ? In particular, is u_t for any fixed t a relative extremal function on D and if not, how far is it from being such?*

Note that

$$\mathbf{E}(\omega_K) = \int_D \omega_K (dd^c \omega_K)^n = - \int_D (dd^c \omega_K)^n = -\text{Cap}(K), \quad (9)$$

the Monge-Ampère capacity of K with respect to D . We have, by Theorem 3.3, the following

Proposition 4.1 *If u_t is the geodesic for a pair of relative extremal functions ω_{K_j} , then*

$$\mathbf{E}(u_t) = (t - 1) \text{Cap}(K_0) - t \text{Cap}(K_1).$$

Denote $L_t = \{z \in D : u_t(z) = -1\}$, then we have $u_t \leq \omega_{L_t}$. By Corollary 2.2 and Proposition 4.1, this implies

Proposition 4.2 *In the conditions of Proposition 4.1,*

$$\text{Cap}(L_t) \leq (1 - t) \text{Cap}(K_0) + t \text{Cap}(K_1),$$

and the inequality becomes equality if and only if ω_{L_t} is the geodesic.

Now let us assume D to be a bounded complete logarithmically convex Reinhardt domain of \mathbb{C}^n , that is, $y \in D$ provided $z \in D$ and $|y_l| \leq |z_l|$ for all l , and such that the set $\log D = \{s \in \mathbb{R}_+^n : \text{Exp } s \in D\}$ is a convex subset of \mathbb{R}^n ; here $\text{Exp } s = (e^{s_1}, \dots, e^{s_n})$. In

addition, let K_j , $j = 0, 1$, be compact Reinhardt subsets of D . In this setting, the functions ω_{K_j} are toric (multi-circled) and so, the function

$$\check{u}(s, t) := u_t(\text{Exp } s)$$

is convex in $(s, t) \in \mathbb{R}^n \times (0, 1)$. Denote

$$K_t = K_0^{1-t} K_1^t = \{z \in \mathbb{D}^n : |z_l| = |\eta_l|^{1-t} |\xi_l|^t, \ 1 \leq l \leq n, \ \eta \in K_0, \ \xi \in K_1\}, \quad 0 < t < 1; \quad (10)$$

in other words, $\log K_t = (1-t) \log K_0 + t \log K_1$. Note that $K_t \subset D$ because $\log D$ is convex.

Recall that volumes $|\cdot|$ of convex combinations $(1-t)P_0 + tP_1$ of two bodies $P_j \subset \mathbb{R}^n$ satisfy

$$|(1-t)P_0 + tP_1| \geq |P_0|^{1-t} |P_1|^t,$$

the Brunn-Minkowski inequality (in multiplicative form). In our case, the sets $\log K_j$ typically are of infinite volume. Instead of the volumes, we have a *reversed Brunn-Minkowski inequality for the capacities* of K_t (multiplicative combinations of K_j), in additive form.

Theorem 4.3 *In the Reinhardt situation, the capacities of the sets K_t defined by (10) satisfy*

$$\text{Cap}(K_t) \leq (1-t) \text{Cap}(K_0) + t \text{Cap}(K_1).$$

Proof. By the convexity of \check{u} , we have $\check{u}(s, t) \leq -1$ when $s \in (1-t) \log K_0 + t \log K_1 = \log K_t$. Therefore, $K_t \subset L_t$, and the result follows from Proposition 4.2. \square

Evidently, ω_{K_t} is the geodesic if and only if $\check{\omega}_{K_t}(s)$ is convex in (s, t) . It turns out that the latter need not be true.

Example 4.4 *Let $n = 1$, $D = \mathbb{D}$, $K_0 = \{z : |z| \leq e^{-1}\}$ and $K_1 = \{z : |z| \leq e^{-2}\}$. Then $K_t = \{z : |z| \leq e^{-1-t}\}$ and the function*

$$\check{\omega}_{K_t}(s) = \max \left\{ \frac{s}{1+t}, -1 \right\}$$

is not convex in (s, t) , so ω_{K_t} is not geodesic. It is easy to check that

$$\check{u}(s, t) = \max \left\{ s, \frac{s+t-1}{2}, -1 \right\},$$

so $K_t = L_t$ and u_t is not a relative extremal function at all.

Note also that $\mathbf{E}(\omega_{K_t}) = -\text{Cap}(K_t) = -(1+t)^{-1}$ is far from being linear. Finally, $\mathbf{E}(u_t) = t/2 - 1$, as expected.

In this example, the geodesics u_t still pertain some features of relative extremal functions. Namely, recall that a *pluriregular condenser* $(K_1, \dots, K_m, \sigma_1, \dots, \sigma_m)$ is a system of pluriregular compact sets $K_m \subset K_{m-1} \subset \dots \subset K_1 \subset D \subset \overline{D} = K_0$ and numbers $\sigma_m < \sigma_{m-1} < \dots < \sigma_1 < \sigma_0 = 0$ such that there is a continuous plurisubharmonic function ω on D with zero boundary values, $K_i = \{z \in D : \omega \leq \sigma_i\}$ and ω is maximal on the complement of K_i in the interior of K_{i-1} , see [20]. In our case, u_t is the extremal function for the condenser $(K_{1,t}, K_{2,t}, t-1, -1)$, where $K_{1,t} = \{z : |z| \leq e^{1-t}\}$ and $K_{2,t} = \{z : |z| \leq e^{-1-t}\}$, and $\mathbf{E}(u_t)$ is the energy of the condenser.

It would be nice to know if anything similar holds in the general case of geodesics of relative extremal functions.

5 Geodesics on \mathcal{F}_1

One cannot apply the above construction to functions from $\mathcal{F}_1(D)$ directly, because they need not be bounded from below and thus existence of the 'good' envelope \widehat{v} is not guaranteed (in the next section, we will show that generally there are no geodesics for plurisubharmonic functions with nonzero Lelong numbers).

Let $u_j \in \mathcal{F}_1(D)$, $j = 0, 1$ and let $u_{j,N} \in \mathcal{E}_0(D)$ decrease to u_j as $N \rightarrow \infty$. Then their geodesics $u_{t,N} \in \mathcal{E}_0(D)$ linearize the functional \mathbf{E} :

$$\mathbf{E}(u_{t,N}) = (1-t) \mathbf{E}(u_{0,N}) + t \mathbf{E}(u_{1,N}).$$

Since $u_{t,N} \geq u_1 + u_2 \in \mathcal{F}_1(D)$ for any N , the functions $u_{t,N}$ decrease to functions $v_t \in \mathcal{F}_1(D)$ and $\mathbf{E}(u_{t,N})$ decrease to $\mathbf{E}(v_t)$ for $0 < t < 1$ while $\mathbf{E}(u_{j,N})$ decrease to $\mathbf{E}(u_j)$ for $j = 0, 1$ by [7, Thm. 3.8]. Therefore,

$$\mathbf{E}(v_t) = (1-t) \mathbf{E}(u_0) + t \mathbf{E}(u_1). \quad (11)$$

Nor also that since $\widehat{u}_N(z, \zeta) = u_{\log|\zeta|, N}(z)$ satisfy $(dd^c \widehat{u}_N)^{n+1} = 0$ on $D \times S$ and decrease to $\widehat{v}(z, \zeta)$, we have $(dd^c \widehat{v})^{n+1} = 0$ as well.

To have a complete analogy with the bounded case, we need to establish the relations $v_t \rightarrow u_j$ as $t \rightarrow j$ for $j = 0$ and 1 . Since v_t are convex in t and $v_t \geq u_1 + u_2$, the functions $v_j = \limsup_{t \rightarrow j} v_t$ are weak limits of v_t and belong to $\mathcal{F}_1(D)$. By construction, $v_j \leq u_j$.

Denote $V_t = (1-t)v_0 + tv_1$. Then a direct computation shows $\mathbf{E}(V_t) \rightarrow \mathbf{E}(v_j)$ as $t \rightarrow j$. Since $v_t \leq V_t$, we get

$$\mathbf{E}(u_j) = \lim_{t \rightarrow j} \mathbf{E}(v_t) \leq \lim_{t \rightarrow j} \mathbf{E}(V_t) = \mathbf{E}(v_j),$$

which implies $u_j = v_j$ by Corollary 2.2.

So, from now on we rename the functions v_t to u_t since they have u_j as there endpoints, for the moment as upper limits when $t \rightarrow j$. We claim that actually $u_t \rightarrow u_j$ in capacity, that is, for any $\epsilon > 0$, we have $\text{Cap } A_{\epsilon,t} \rightarrow 0$, where $A_{\epsilon,t} = A_{\epsilon,t,j} = \{z \in D : |u_t(z) - u_j(z)| > \epsilon\}$.

We will prove the claim for $j = 0$, the case $j = 1$ being completely similar. By subadditivity of the capacity, it suffices to show that $\text{Cap } A_{\epsilon,t}^\pm \rightarrow 0$, where $A_{\epsilon,t}^+ = \{z : u_t(z) - u_0(z) > \epsilon\}$ and $A_{\epsilon,t}^- = \{z : u_t(z) - u_0(z) < -\epsilon\}$. Moreover, since $u_t \geq u_0 + u_1$ and $\text{Cap } \{z : u_0(z) + u_1(z) < -N\} = o(N^{-n})$ as $N \rightarrow \infty$ [2, Lemma 2.1], we can assume $u_0 + u_1 \geq -N$ on $A_{\epsilon,t}$.

Since $u_t \leq V_t$, we have $u_1 - u_0 \geq \epsilon/t$ on $A_{\epsilon,t}^+$, so for any $\psi \in \text{PSH}(D)$, $-1 \leq \psi \leq 0$,

$$\int_{A_{\epsilon,t}^+} (dd^c \psi)^n \leq t \epsilon^{-1} \int_{A_{\epsilon,t}^+} (u_1 - u_0) (dd^c \psi)^n \leq t N \epsilon^{-1} \int_{A_{\epsilon,1/2}^+} (dd^c \psi)^n \leq t N \epsilon^{-1} \text{Cap } A_{\epsilon,1/2}^+$$

for any $t < 1/2$, which implies $\text{Cap } A_{\epsilon,t}^+ \rightarrow 0$ as $t \rightarrow 0$.

To work with the set $A_{\epsilon,t}^-$ is more tricky because, in the unbounded case, there are no straightforward subgeodesics with good behavior at the endpoints. We will use here an envelope technique introduced (in the Kähler setting) in [22] and developed in [11] (especially in Theorems 4.3 and 5.2 of the latter paper).

Given $u, v \in \text{PSH}(D)$, denote the largest plurisubharmonic minorant of $\min\{u, v\}$ in D by $P(u, v)$. If $u_0, u_1 \in \mathcal{F}_1(D)$ and $C \geq 0$, then

$$u_0 + u_1 \leq p_C := P(u_0, u_1 + C) \leq u_0,$$

which implies $p_C \in \mathcal{F}_1(D)$. Therefore, $w_{t,C} = p_C - Ct \leq u_t$ and

$$A_{\epsilon,t}^- \subset \{z : w_{t,C}(z) - u_0(z) < -\epsilon\}.$$

Therefore,

$$\lim_{t \rightarrow 0} \text{Cap } A_{\epsilon,t}^- \leq \inf_{C \geq 0} \text{Cap } B_{\epsilon,C}^-,$$

where $B_{\epsilon,C}^- = \{z : p_C(z) - u_0(z) < -\epsilon\}$.

The family p_C increases in C to a function whose upper semicontinuous regularization is a plurisubharmonic function U . Moreover, p_C converges to U in capacity, so our claim follows from the lemma below.

Lemma 5.1 *For any $u_0, u_1 \in \mathcal{F}_1(D)$, let $p_C := P(u_0, u_1 + C)$ and let U be the upper semicontinuous regularization of $\lim_{C \rightarrow \infty} p_C$. Then $U = u_0$ on D .*

Proof. Our arguments are close to the proof of [11, Thm. 4.3]. Precisely as for ω -plurisubharmonic functions in [11, Prop. 3.3], we have for any bounded plurisubharmonic functions u and v the inequality

$$(dd^c P(u, v))^n \leq \mathbf{1}_{\{P=u\}}(dd^c u)^n + \mathbf{1}_{\{P=v\}}(dd^c v)^n.$$

Let $u_{0,N}, u_{1,N} \in \mathcal{E}_0(D)$ be approximations of $u_0, u_1 \in \mathcal{F}_1(D)$ from the definition of the class \mathcal{F}_1 , and set $u = u_{0,N}, v = u_{1,N} + C$ for $C > 0$. Then

$$\mathbf{1}_{\{P=u\}}(dd^c u)^n \leq (dd^c u_{0,N})^n$$

and

$$\mathbf{1}_{\{P=v\}}(dd^c v)^n \leq \mathbf{1}_{\{u_{1,N} < -C\}}(dd^c u_{1,N})^n.$$

For any positive test function $\eta \in C_0(D)$, we have

$$\begin{aligned} \int_D \eta \mathbf{1}_{\{u_{1,N} < -C\}}(dd^c u_{1,N})^n &\leq \frac{1}{C} \int_D \eta |u_{1,N}| \mathbf{1}_{\{u_{1,N} < -C\}}(dd^c u_{1,N})^n \\ &\leq \frac{\max \eta}{C} |\mathbf{E}(u_{1,N})| \rightarrow \frac{\max \eta}{C} |\mathbf{E}(u_1)| \end{aligned}$$

as $N \rightarrow \infty$. Since $P(u, v)$ decreases to p_C as $N \rightarrow \infty$, we get then

$$\int_D \eta (dd^c p_C)^n \leq \int_D \eta (dd^c u_0)^n + \frac{\max \eta}{C} |\mathbf{E}(u_1)|,$$

and with $C \rightarrow \infty$ we deduce $(dd^c U)^n \leq (dd^c u_0)^n$. Both U and u_0 belong to $\mathcal{F}_1(D)$, so this implies, by [7, Thm. 4.5], $U \geq u_0$. Since $U \leq u_0$, the proof of the lemma is complete. \square

We summarize the results of this section as follows.

Theorem 5.2 *For any pair $u_0, u_1 \in \mathcal{F}_1(D)$ there exists a geodesic $u_t \in \mathcal{F}_1(D)$, $0 < t < 1$, such that u_t converge in capacity to u_j as t approaches $j = 0$ and $j = 1$. The energy functional $v \mapsto \mathbf{E}(v)$ is concave on $\mathcal{F}_1(D)$, while the function $t \mapsto \mathbf{E}(u_t)$ is linear on geodesics u_t and convex on subgeodesics $v_t \in \mathcal{F}_1(D)$.*

Corollary 5.3 *The uniqueness result of Theorem 3.4 remains true for $u_0, u_1 \in \mathcal{F}_1(D)$.*

6 Case of strong singularities

The Monge-Ampère current $(dd^c u)^n$ of functions from the class \mathcal{F}_1 cannot charge pluripolar sets. If functions $u_j \in \text{PSH}^-(D)$ are allowed to have stronger singularities, the process of constructing geodesics generally fails. The breaking point is that the presumed 'geodesic' u_t can have $\lim_{t \rightarrow j} u_t < u_j$.

We start with a simple observation. Let $a \in D$ and let G_a be the pluricomplex Green function of D with pole at a .

Lemma 6.1 *If $\Phi \in \text{PSH}^-(D \times S)$ is such that $\limsup \Phi(z, \zeta) \leq G_a(z)$ for all $z \in D$ as $|\zeta| \rightarrow e$, then $\Phi(z, \zeta) \leq G_a(z)$ for all $z \in D$ and all $\zeta \in S$.*

Proof. The functions $\psi_N(z, \zeta) = \max\{G_a(z), -N \log |\zeta|\} \in \text{PSH}^-(D \times S)$ are equal to 0 on $\partial D \times S$. We also have $\psi_N(z, \zeta) \rightarrow u_{N,0}(z) = 0$ when $|\zeta| \rightarrow 1$, and $\psi_N(z, \zeta) \rightarrow u_{N,1}(z) = \max\{G_a(z), -N\}$ when $|\zeta| \rightarrow e$.

Furthermore, they satisfy $(dd^c \psi_N)^{n+1} = 0$ everywhere in $D \times S$. Therefore, $\psi_{N,t}$ is the geodesic for $u_{N,0}$ and $u_{N,1}$. Since $\Phi \leq \psi_N$ for any N , the proof is complete. \square

A bit more generally, let $u \in \text{PSH}^-(D)$ be such that $A = \{z : u(z) = -\infty\}$ is a closed subset of D and $u \in L_{loc}^\infty(D \setminus A)$. Then the function

$$g_u(z) = \limsup_{x \rightarrow z} \sup\{v(x) : v \in \text{PSH}^-(D), v \leq u + O(1)\}$$

is plurisubharmonic in D , locally bounded outside A and satisfying $(dd^c g_u)^n = 0$ there. When A is a single point, then $g_u \equiv 0$ if and only if $(dd^c u)^n(A) = 0$ [21].

As is easy to see, $g_u \not\equiv 0$ if u has nonzero Lelong number at some point of A ; we do not know if the converse is true.

By repeating the arguments of the proof of Lemma 6.1, we get

Theorem 6.2 *If $\Phi \in \text{PSH}^-(D \times S)$ is such that*

$$\limsup_{\log |\zeta| \rightarrow j} \Phi(z, \zeta) \leq u_j(z) \quad \forall z \in D, \quad j = 0, 1, \quad (12)$$

then $\Phi(z, \zeta) \leq P(z)$ for all $\zeta \in S$, where $P = P(g_{u_0}, g_{u_1})$ is the largest plurisubharmonic minorant of the function $\min_j g_{u_j}$. In particular, if each $u_j = g_{u_j}$, then the largest Φ satisfying (12) coincides with P (and thus is independent of ζ .)

Example 6.3 *Let A be a finite subset of D and let u_j equal the multi-pole Green function of A with weights $m_{j,k} \geq 0$ at $a_k \in A$. Then the best function Φ satisfying (12) is the multi-pole Green function of A with weights $M_k = \max_j m_{j,k}$ at $a_k \in A$.*

Remark. The situation changes if one replaces the segment $0 < t < 1$ with the ray $-\infty < t < 0$. For example, let $\varphi_j = u_j + w_j$ such that $u_j \in \mathcal{E}_1(D)$ and $w_0 = w_1 + w$, where $w \in \text{PSH}^-(D)$ has zero boundary values. If u_t , $0 < t < 1$, is the geodesic arc for u_0 and u_1 , then

$$\varphi_t = u_{e^t} + w_1 + \max\{w, t\}, \quad -\infty < t < 0,$$

is a subgeodesic ray with $\varphi_t \rightarrow \varphi_j$ as $t \rightarrow \log j$, $j = 0, 1$.

7 Relations to the Kähler case

Let (X, ω) be a compact Kähler manifold. An upper semicontinuous function φ on X is called ω -plurisubharmonic if $\omega + dd^c\varphi \geq 0$. Cegrell's classes were generalized to such functions in [17]. A corresponding class $\mathcal{E}_1(X, \omega)$ was introduced, and it has turned to be a natural frame for studying the Mabuchi functional [3]; see also a nice presentation in [16], where, in addition, toric geodesics on toric manifolds are considered.

Some of problems studied in recent papers by T. Darvas with co-authors (e.g., [4], [11], [12], [13]) in the Kähler setting are close to those treated here. In particular, Proposition 4.2 from [4] is a complete analog of our Corollary 2.2. Theorem 5.2 from [11] characterizes ω -plurisubharmonic functions that can be joined by a weak geodesic in terms of a technique from [22], which is closely related to our Theorem 6.2. Finally, we have borrowed the idea of using the envelope technique for proving convergence in capacity in Theorem 5.2 from Theorems 4.3 and 4.3 of [11].

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References

- [1] R. BERMAN AND S. BOUCKSOM, *Growth of balls of holomorphic sections and energy at equilibrium*, Invent. Math. **181** (2010), no. 2, 337–394.
- [2] S. BENELKOURCHI, *Weighted pluricomplex energy*, Potential Anal. **31**(2009), no. 1, 1–20.
- [3] R. BERMAN, S. BOUCKSOM, V. GUEDJ, A. ZERIAHI, *A variational approach to complex Monge-Ampère equations*, Publ. Math. Inst. Hautes Études Sci. **117** (2013), 179–245.
- [4] R. BERMAN, T. DARVAS, CHINH H. LU, *Convexity of the extended K-energy and the large time behaviour of the weak Calabi flow*, <http://arxiv.org/abs/1510.01260>.
- [5] B. BERNDTSSON, *A Brunn-Minkowski type inequality for Fano manifolds and some uniqueness theorems in Kähler geometry*, Invent. Math. **200** (2015), no. 1, 149–200.
- [6] U. CEGRELL, *Capacities in complex analysis. Aspects of Mathematics*, E14. Friedr. Vieweg & Sohn, Braunschweig, 1988.
- [7] U. CEGRELL, *Pluricomplex energy*, Acta Math. **180** (1998), no. 2, 187–217.
- [8] U. CEGRELL, *The general definition of the complex Monge–Ampère operator*, Ann. Inst. Fourier (Grenoble) **54** (2004), no. 1, 159–179.
- [9] U. CEGRELL, *Approximation of plurisubharmonic functions in hyperconvex domains*, Complex analysis and digital geometry, 125–129, Acta Univ. Upsaliensis Skr. Uppsala Univ. C Organ. Hist., 86, Uppsala Universitet, Uppsala, 2009.
- [10] X.X. CHEN, *The space of Kähler metrics*, J. Diff. Geom. **56** (2000), no. 2, 189–234.

- [11] T. DARVAS, *The Mabuchi Completion of the Space of Kähler Potentials*, <http://arxiv.org/abs/1401.7318>.
- [12] T. DARVAS, *The Mabuchi Geometry of Finite Energy Classes*, <http://arxiv.org/abs/1409.2072>.
- [13] T. DARVAS AND Y. RUBINSTEIN, *Kiselman's principle, the Dirichlet problem for the Monge-Ampere equation, and rooftop obstacle problems*, <http://arxiv.org/abs/1405.6548>.
- [14] S.K. DONALDSON, *Symmetric spaces, Kähler geometry and Hamiltonian dynamics*, Northern California Symplectic Geometry Seminar, 13–33, Amer. Math. Soc. Transl. Ser. 2, **196**, Amer. Math. Soc., Providence, RI, 1999.
- [15] Complex Monge-Ampere equations and geodesics in the space of Kähler metrics. Edited by V. Guedj. Lecture Notes in Math., **2038**, Springer, 2012.
- [16] V. GUEDJ, *The metric completion of the Riemannian space of Kähler metrics*, <http://arxiv.org/abs/1401.7857>.
- [17] V. GUEDJ AND A. ZERIAHI, *The weighted Monge-Ampere energy of quasisubharmonic functions*, J. Funct. Anal. **250** (2007), no. 2, 442–482.
- [18] P. GUAN, *The extremal functions associated to intrinsic metrics*, Ann. Math. (2) **156** (2002), 197–211.
- [19] T. MABUCHI, *Some symplectic geometry on compact Kähler manifolds. I*, Osaka J. Math. **24** (1987), no. 2, 227 – 252.
- [20] E. POLETSKY, *Approximation of plurisubharmonic functions by multipole Green functions*, Trans. Amer. Math. Soc. **355** (2003), no. 4, 1579–1591.
- [21] A. RASHKOVSKII, *Relative types and extremal problems for plurisubharmonic functions*, Int. Math. Res. Not., 2006, Art. ID 76283, 26 pp.
- [22] J. ROSS AND D. WITT NYSTRØM, *Analytic test configurations and geodesic rays*, J. Symplectic Geom. **12** (2014), no. 1, 125–169.
- [23] S. SEMMES, *Complex Monge-Ampère and symplectic manifolds*, Amer. J. Math. **114**:3, (1992), 495–550.

Tek/Nat, University of Stavanger, 4036 Stavanger, Norway
 E-MAIL: alexander.rashkovskii@uis.no